The [illegible] length [illegible]
curve in [illegible]
of the curve [illegible]
at time [illegible]
point) then that parameterization
has unit speed

Ex. Reparameterize $\vec{r}(t) = \langle 3\sin(t), 2t, 3\cos(t) \rangle$ by arc length (from $t=0$).

Sol: First, we compute the arc length of the function

$$s(t) = \int_0^t |\vec{r}\,'(q)|\,dq \qquad \vec{r}\,'(t) = \langle 3\cos(t), 2, -3\sin(t) \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{9\cos^2(t) + 4 + 9\sin^2(t)} = \sqrt{9+4} = \sqrt{13}$$

$$= \int_{q=0}^{t} \sqrt{13}\,dq$$

$$= \left[\sqrt{13}\,t\right]_0^t = \sqrt{13}\,t - 0 = \sqrt{13}\,t$$

So $s(t) = \sqrt{13}\,t$

$t = \frac{s}{\sqrt{13}}$

Finally, our reparameterized function is $\vec{p}(s) = \vec{r}(t(s))$

$= \left\langle 3\sin\left(\frac{s}{\sqrt{13}}\right), \frac{2s}{\sqrt{13}}, 3\cos\left(\frac{s}{\sqrt{13}}\right) \right\rangle$

NB: For the curve above, $\vec{p}\,'(s) = \left\langle \frac{3}{\sqrt{13}}\cos\left(\frac{s}{\sqrt{13}}\right), \frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}\sin\left(\frac{s}{\sqrt{13}}\right) \right\rangle$

$\therefore$ the magnitude of $\vec{p}\,'(s) = \sqrt{\left(\frac{3}{\sqrt{13}}\cos\left(\frac{s}{\sqrt{13}}\right)\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2 + \left(-\frac{3}{\sqrt{13}}\sin\left(\frac{s}{\sqrt{13}}\right)\right)^2}$

$$= \sqrt{\frac{9}{13}\cos^2\left(\frac{s}{\sqrt{13}}\right) + \frac{4}{13} + \frac{9}{13}\sin^2\left(\frac{s}{\sqrt{13}}\right)}$$

$$= \sqrt{\frac{9}{13} + \frac{4}{13}} = \sqrt{\frac{13}{13}} = 1 \quad \text{for all } s$$

Hence, this reparameterized curve has unit speed. In general, a curve parameterized by arc length always has unit speed

Now, for some physics

Ex. Find the velocity and acceleration of $\vec{r}(t) = \langle 2^t,$ $\qquad 2^t = e^{\ln(2)t}$

$t^2, \ln(t+1)\rangle$ at $t=1$

Sol: $\vec{v}(t) = \vec{r}\,'(t)$

$$= \langle \ln(2) e^{\ln(2)t}, 2t, \tfrac{1}{1+t}\rangle = \langle \ln(2)2^t, 2t, \tfrac{1}{1+t}\rangle$$

So at $t=1$, $\vec{v} = \langle \ln(2)2^1, 2(1), \frac{1}{1+1}\rangle$

$$= \langle 2\ln(2), 2, \tfrac{1}{2}\rangle$$

$$\vec{a}(t) = \vec{r}\,''(t) = \vec{v}\,'(t) = \langle \ln(2)^2 e^{\ln(2)t}, 2, -\tfrac{1}{(1+t)^2}\rangle$$

$$= \langle \ln(2)^2 2^t, 2, -(1+t)^{-2}$$

$$\vec{a}(1) = \langle \ln(2)^2(2(1)), 2, -(1+1)^{-2}\rangle$$

$$= \langle 2\ln(2)^2, 2, -\tfrac{1}{4}\rangle$$

Ex. Find the velocity and position functions of the curve with $\vec{a}(t) = \langle \sin(t), 2\cos(t), 6t\rangle$ and $\vec{v}(0) = \langle 0,0,-1\rangle$, $\vec{r}(0) = \langle 0,-1,-4\rangle$

Sol: $\vec{v}(t) = \int \vec{a}(t)\,dt$

$$= \langle -\cos(t), 2\sin(t), 3t^2\rangle + \vec{c}$$

Now $\langle 0,0,-1\rangle = \vec{v}(0) = \langle -\cos 0, 2\sin 0, 3(0)^2\rangle + \vec{c}$

$$= \langle -1, 0, 0\rangle + \vec{c}$$

$\therefore \vec{c} = \langle 0,0,-1\rangle - \langle -1,0,0\rangle = \langle 1,0,-1\rangle$

$\therefore \vec{v} = \langle -\cos(t), 2\sin(t), 3t^2\rangle + \langle 1,0,-1\rangle$

$$= \langle 1-\cos(t), 2\sin(2), 3t^2-1\rangle$$

Now $\vec{r}(t) = \int \vec{v}(t)\,dt$

$$= \langle t-\sin(t), -2\cos(t), t^3-t\rangle + \vec{c}$$

$$\vec{r}(0) = \langle 0-\sin 0, -2\cos 0, 0^3-0\rangle + \vec{c}$$

$\langle 0,1,-4\rangle = \langle 0,-2,0\rangle + \vec{c}$

$\vec{c} = \langle 0,3,-4\rangle$

$\vec{r}(t) = \langle t-\sin(t), 3-\cos(t), t^3-t-4\rangle$

Ex. When is the speed of particle with position function $\vec{r}(t) = \langle t^2, 5t, t^2-16t \rangle$ at a minimum?

Sol: $\vec{r}\,'(t) = \langle 2t, 5, 2t-16 \rangle$

$f(t) = |\vec{r}\,'(t)| = \sqrt{2t^2 + 5^2 + (2t-16)^2}$

$f(t) = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256}$

$= (8t^2 - 64t + 281)^{1/2}$

$\therefore f'(t) = \frac{1}{2}(8t^2 - 64t + 281)^{-1/2}(16t - 64)$

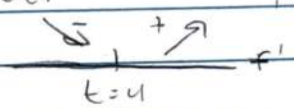
$= \dfrac{8t-32}{(8t^2-64t+281)^{1/2}}$

Note that $64^2 - 4 \cdot 8 \cdot 281 = 2^{12} - 2^5 \cdot 281 = 2^{12} - 2^5 \cdot 256 = 2^{12} - 2^5 \cdot 2^8 = 2^{12} - 2^{13} < 0$

$\therefore 8t^2 - 64t + 281 = 0$ has no real solutions

$\therefore$ the only critical point of this function is at $8t - 32 = 0$ i.e. $t = 4$

Now applying the 1st deriv test, if $f'(t) < 0$ on $t < 4$ and $f'(t) > 0$ on $t > 4$, then $t = 4$ corresponds to a minimum



Now $f'(0) = \frac{-32}{\sqrt{281}} < 0$ and $f'(5) = \frac{8}{(+)} > 0$

Hence the particle is slowest at $t = 4$

Recall: If $f(t) \geq 0$ for all t and f is diff. for all t, then f is minimized exactly when $(f(t))^2$ is minimized

Alt. Sol: $f(t) = |\vec{r}\,'(t)| = (8t^2 - 64t + 281)^{1/2}$ as before

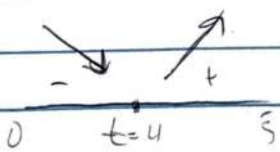
Now we minimize $(f(t))^2 = 8t^2 - 64t + 281$

As before, $8t^2 - 64t + 281 \neq 0$ for all t

Now $(f(t))^2 = g(t)$ is minimized via the 1st deriv test

$g'(t) = 16t - 64$

$g'(t) = 0$ if $t = 4$



$\therefore$ the particle speed is minimized at $t = 4$

Ex. A ball is kicked at angle 60° above ground. If it lands 90m away, at what speed was it thrown? Accel due to grav = 9.8m/s²

Sol:

$a(t) = \langle 0, -9.8 \rangle$

$\vec{v}(0) = |\vec{v}(0)| \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle = c\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$\vec{r}(t_0) = \langle 90, 0 \rangle$

$\therefore \vec{v}(t) = \int \vec{a}(t)\, dt$

$= \langle \alpha, -9.8t + \beta \rangle$

$\vec{v}(0) = \frac{c}{2}\langle 1, \sqrt{3} \rangle$

$\therefore \begin{cases} \alpha = \frac{c}{2} \\ \beta = \frac{\sqrt{3}}{2}c \end{cases}$ $\quad \therefore v(t) = \langle \frac{c}{2}, -9.8t + \frac{\sqrt{3}}{2}tc \rangle$

$\vec{r}(t) = \int \vec{v}(t)\, dt$

$= \langle \frac{c}{2}t + \gamma, -4.9t^2 + \frac{\sqrt{3}}{2}c + \delta \rangle$

Now at some time t_0 we have

$\vec{r}(t_0) = \langle 90, 0 \rangle = \langle \frac{c}{2}t_0 + \gamma, -4.9t_0^2 + \frac{\sqrt{3}}{2}ct_0 + \delta \rangle$

We may assume $\vec{r}(0) = \langle 0, 0 \rangle$

Now notice with our assumption $\vec{r}(0) = \langle 0, 0 \rangle$ we obtain $\langle \gamma, \delta \rangle = 0$

$r(t_0) = \langle \frac{c}{2}t_0, -4.9t_0^2 + \frac{\sqrt{3}}{2}c \rangle$ $\quad \therefore \frac{c}{2}t_0 = 90$ so $t_0 = \frac{180}{c}$

$\therefore -4.9\left(\frac{180}{c}\right)^2 + \frac{\sqrt{3}}{2}c = 0$

Ex. A ball is kicked at an[illegible] [illegible] if it lands 90m away, at what speed was it thrown [illegible]

Sol.

$\vec{a}(t) = \langle 0, -9.8 \rangle$

$\vec{v}(0) = |\vec{v}(0)| \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle = c\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$\vec{r}(t_0) = \langle 90, 0 \rangle$

$\therefore \vec{v}(t) = \int \vec{a}(t)\,dt$

$= \langle \alpha, -9.8t + \beta \rangle$

$\vec{v}(0) = \frac{c}{2} \langle 1, \sqrt{3} \rangle$

$\therefore \begin{cases} \alpha = \frac{c}{2} \\ \beta = \frac{\sqrt{3}}{2}c \end{cases}$ $\quad \therefore v(t) = \langle \frac{c}{2}, -9.8t + \frac{\sqrt{3}}{2}c \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt$

$= \langle \frac{c}{2}t + \gamma, -4.9t^2 + \frac{\sqrt{3}}{2}ct + \delta \rangle$

Now at some time t_0 we have

$\vec{r}(t_0) = \langle 90, 0 \rangle = \langle \frac{c}{2}t_0 + \gamma, -4.9t_0^2 + \frac{\sqrt{3}}{2}ct_0 + \delta \rangle$

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Now notice with our assumption $\vec{r}(0) = \langle 0, 0 \rangle$ we obtain $\langle \gamma, \delta \rangle = 0$

$r(t_0) = \langle \frac{c}{2}t_0, -4.9t_0^2 + \frac{\sqrt{3}}{2}ct_0 \rangle \quad \therefore \frac{c}{2}t_0 = 90$ so $t_0 = \frac{180}{c}$

$\therefore -4.9\left(\frac{180}{c}\right)^2 + \frac{\sqrt{3}}{2}c \cdot \frac{180}{c} = 0$